SOME GAME THEORY MODELS FOR ALLOCATING FORCES IN A NARROW STRAIT AGAINST HOSTILE ACTIVITY

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THESIS

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Some Game Theory Models for Allocating Forces

in a Narrow Strait against Hostile Activity

by

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ABSTRACT

The purpose of this thesis is to summarize some game theoretical models which can be applied to situations of conflict between two opponents.

Opponent one's objective is to guard a long and narrow strait, with his high speed boats, against the other who tries to cross.

The models considered are zero sum game models, games with different payoff functions and a game with deadline where the game must be terminated in a predetermined period.

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I. INTRODUCTION

The purpose of this thesis is to summarize some game theoretical models for dealing with the following problem.

A. PROBLEM STATEMENT

Consider a long and narrow strait where smuggling or terror activity is taking place. A patrol unit (which shall be called side A) patrols in order to catch the smugglers or terrorists (side B). Side A is equipped with speed boats which have a search radar, communication units and all of its boats are under one command. Side B consists of individuals with small motor boats which are hard to detect. These motor boats do not have communication or radar. They cross the strait at night, usually near their village or near a place where there is a village on the other side. Being near the shore their radar echo is shadowed by land and they are impossible to detect. Therefore, they can travel to a favorite crossing point without being detected and then try to cross.

Although side A's boats are much faster, the fact that the strait is long and narrow gives side B a chance to cross successfully without being detected. Both sides would like to use the "best" strategy. The best strategy for side A is the strategy which will maximize the number of boats captured from side B. Side B views the best strategy as the strategy which maximizes the number of

trips per boat before capture. A measure of effectiveness could be the probability that B's boat makes the trip. Given a measure of effectiveness, side A would like to know what will happen if certain operating conditions are changed. In particular, he is concerned with how his effectiveness and his best strategy are going to change if he changes the number of patrol boats assigned to the strait, or how they would change if he can have boats with different detection capabilities. A would like to know how to take into consideration intelligence and how to determine the value of his intelligence according to the results he achieves. Side B, after determining his measure of effectiveness, would like to know the best strategy and find out when the situation is too risky and therefore should terminate for a period of time.

The fact that side A's boats are much faster and detection of one of B's boats can be made only if it is far away from shore enables A to catch B whenever he detects them. We assume that there is not much traffic in the strait and therefore when side A detects a boat, he always captures it. Thus we shall consider the probability of detection as a measure of effectiveness for side A.

B. FORMULATION OF THE PROBLEM

In order to be able to determine a best strategy it is convenient to define some mathematical models which will describe the situation. We partition the strait into N imaginary strips as shown in Figure 1.

In this paper we do not deal with the problem of selecting the optimal number of strips. In practice it will be a function of computation ability, the accuracy desired and the boat's detection and speed capabilities.

It is assumed that within each strip the probability that a patrol boat from A detecting one of B's boats is constant. Computing these probabilities is not an easy task since they depend upon such factors as radar characteristic, weather conditions, target characteristic, mutual speeds, etc. Some computational methods for this case are presented in Refs. 2, 4, and 5.

C. SCOPE

In trying to come up with the "best" strategy we used a game theory approach. We consider side A and B as two opponents and assign payoff from B to A. In chapter 2 and 3 we associate the payoff with the conditional probability that side A catches B. We assign a payoff of 1 if A catches B, when B decides to cross, and 0 if B crosses successfully. Therefore, the value of the game is the conditional expectation that A will catch B when B decides to cross. Note that it is senseless to use the unconditional expected number of catches as the measure of effectiveness since B has the option of not crossing at all.

Chapter 4 deals with games with deadline where side B must cross the strait during a certain period or he loses

the game. In this case we were able to relate the payoff to the probability that A catches B rather than to the conditional probability as before.

Chapter 4 defines the payoff from B to A as +1 if A catches B and -1 if B crosses successfully. As a result we get a recursive game and we find the value of the game by solving a difference equation. The value of this game can be considered as a measure of effectiveness. In order to get the value of this game one has to know the conditional expectations. These expectations can be obtained by one of the models of chapter 2 or 3.

II. ZERO SUM GAME

A. INTRODUCTION

In order to be able to formulate this conflict as a zero sum game we must assume that both opponents have strictly opposite preferences. Side A strictly prefers catching boats from side B and B prefers the opposite. Side A's pure strategy is to allocate boats to a particular set of strips. Side B's pure strategy is to cross in one particular strip.

For any pure strategy of A and any pure strategy for B we define:

P_{ij} = Probability that A will detect B, when A chooses to use his i th pure strategy and B uses his j th.

We can consider P_{ij} as the payoff to A, and by the assumption of strictly opposite preference it is the loss to B. Assuming that the game is played over and over again, a mixed strategy is the proportion according to which each player uses his pure strategies. In order to be able to solve the game we must assume further that each player wants to maximize his total payoff. Thus we can say that each player tries to maximize his expected payoff function. Since the number of pure strategies is finite, the sets of mixed strategies are closed and bounded subsets of Euclidean space and the objective function is continuous [Ref. 10].

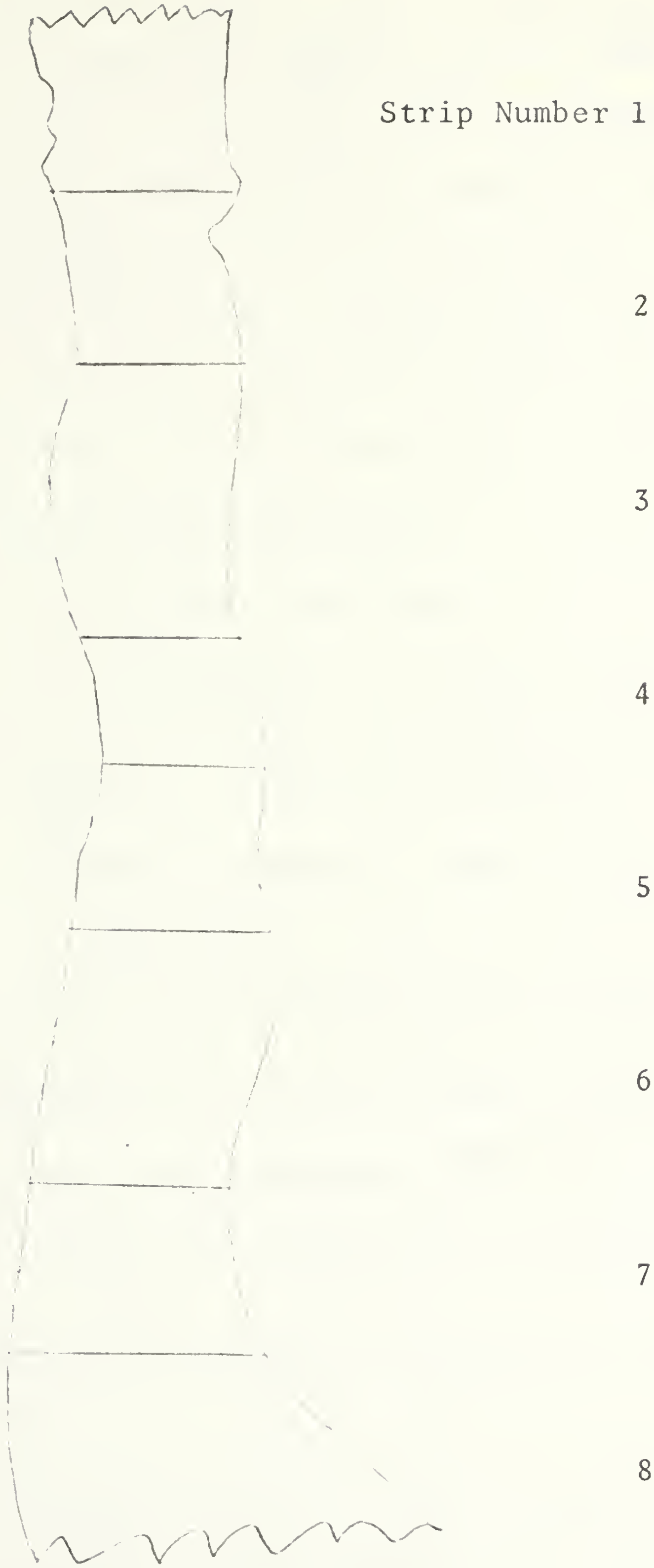


FIGURE 1

These games are known to have at least one equilibrium point and all equilibrium points have the same value, called the value of the game. Since the payoff to A represents the probability that A detects B when both use some pure strategies, the value of the game is the proportion of B's boats that A detects when both use their best mixed strategies. Thus we can define the value of the game as a measure of effectiveness for A and B.

B. ZERO SUM GAME ONE PATROL BOAT

Here we consider the case where

A (patroling force) has only one boat and

B (smugglers) can cross at strips 1, 2. . . . . . . N.

Define the mixed strategies: Let $a = (a_1, \ldots, a_N)$, $b = (b_1, \ldots, b_N)$ where

a_i = probability that side A uses its pure strategy i;

b_j = probability that side B uses its pure strategy j.

If we accept the assumptions made in section A, then by the Minimax Theorem, the value of the game is given by:

$$(1) \quad \max_i \min_j \sum a_i P_{ij} b_j = \min_j \max_i \sum a_i P_{ij} b_j = V.$$

This game is easily solved by linear programming [Ref. 3]. Side A as a player wants to maximize the value of the game. He can do that if he plays a mixed strategy $\bar{a}^*$. By doing

so he will get at least V no matter what B does (Von Newmann-Morgenstern Theorem [Ref. 10]).

In particular, A will get V or more if B uses any of its pure strategies. Thus A wishes to solve the following linear program:

Max V

s.t.

$$
\begin{aligned}
a_1P_{11} + a_2P_{21} + \cdots\cdots\cdots\cdots a_nP_{n1} &\geq V \\
a_1P_{12} + a_2P_{22} + \cdots\cdots\cdots\cdots a_nP_{n2} &\geq V \\
\vdots \qquad\quad \vdots \qquad\qquad\qquad\qquad \vdots \\
a_1P_{1n} + a_2P_{2n} + \cdots\cdots\cdots\cdots +a_nP_{nn} &\geq V
\end{aligned}
$$

$$\sum_{i=1}^{n} a_i = 1$$

Subtracting the first row of the constraint matrix from any other row but the last yields a different form (P1) which is easier to solve.

Max V

(P1) s.t.

$$
\begin{aligned}
a_1(P_{12} - P_{11}) + a_2(P_{22} - P_{21}) \cdot\cdot + a_n(P_{n2} - P_{n1}) &\geq 0 \\
\vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \\
a_1(P_{1n} - P_{11}) + a_2(P_{2n} - P_{21}) + \cdots\cdots a_n(P_{nn} - P_{n1}) &\geq 0
\end{aligned}
$$

$$\sum_{i=1}^{n} a_i = 1$$

Using the same reasoning, we get a similar formulation for B which is the dual of (P1) called (D1).

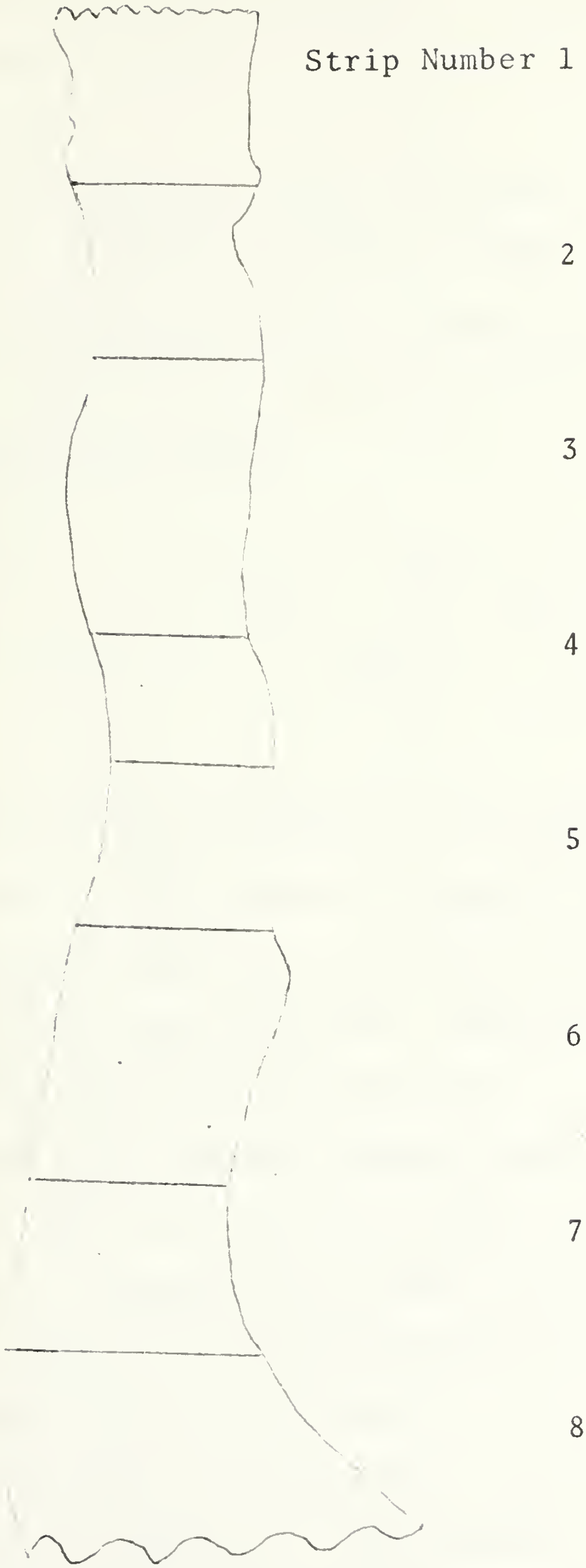
$$
\text{(D1)}\quad \begin{aligned}
&\text{Min } V \\
&\text{s.t.} \\
&(P_{21}-P_{11})b_1 + (P_{22}-P_{12})b_2 + \ldots\ldots (P_{2n}-P_{1n})b_n \le 0 \\
&\quad\vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \\
&(P_{n1}-P_{11})b_1 + (P_{n2}-P_{12})b_2 + \ldots\ldots (P_{nn}-P_{1n}) \le 0 \\
&\sum_{j=1}^{n} b_j = 1
\end{aligned}
$$

C. ZERO SUM GAME, MORE THAN ONE PATROL BOAT

The model in section B can be extended by allowing side A to have n boats. First we extend the definition of pure strategy for A to be a particular allocation of his boats.

For example, if we have 10 strips and side A has three boats a pure strategy is to allocate boat number 1 to strip 1, boat number 2 to strip 3 and boat 3 to strip 5.

If side A had only one boat it would have only N pure strategies. With n boats, however, side A has N^n pure strategies. We can index these pure strategies by $i = 1, 2, \ldots\ldots N^n$; and we define a set ℓ_i which associates with each index i, a particular allocation of boats to particular strips (see Figure 2).



Index i	Boat Number 1	2
1	1	1
2	1	2
3	1	3
4	1	4
5	1	5
6	1	6
7	1	7
8	1	8
9	2	1
10	2	2
⋮		
30	10	8
31	10	9
32	10	10

Strip Number

Figure 2

A pure strategy indexed by i, therefore, refers to an allocation of boats to strips according to ℓ_i.

Define:

$\overline{P_{ij}}$ ≡ Probability side A detects B when A uses his pure strategy indexed by i and B crosses at j.

p_{ij} ≡ Probability that A detects B with a boat assigned to strip k according the set of allocations ℓ_i.

Thus, it follows that

$$P_{ij} = 1 - \prod_{\substack{\text{for all} \\ k \in \ell_i}} (1 - p_{kj}) \tag{2}$$

As in section B this game can be formulated using linear programming. The method of solution remains the same but the simplex algorithm or any other used will take more calculation time because the number of rows is now $N^n + 1$. The number of columns is still N.

Side A has a mixed strategy consisting of N^n elements where side B has a mixed strategy consisting of only N elements. Expanding the number of boats for A does not change the set of strategies for B but it changes considerably the set for side A.

The value of the game, V, is the proportion of B boats which are caught by A. One question of interest is the marginal effect of providing an additional patrol boat

for side A. In order to answer this question one must solve the game with n boats and then with n + 1 boats. This is a painful process since the two games have different payoff matrices and different strategies. Another interesting question is what happens if A's detection capabilities change. The simple, but painful, way is to formulate the problem again and to proceed from the beginning. However, a revised solution can be obtained by making pivots in the old basis from previous solution [Ref. 3]. This is illustrated in the following example.

1. Example 1

Consider a strait 100 miles long with 10 possible crossing points. Side A has only one boat operating against B with detection capabilities shown in Table 1.

TABLE 1

A patrol at \ B cross at	1	2	3	4	5	6	7	8	9	10
1	.8	.6	.2							
2	.6	.8	.6	.2						
3	.2	.6	.8	.6	.2					
4		.2	.6	.8	.6	.1				
5			.1	.6	.8	.5	.1			
6				.1	.6	.8	.5			
7						.5	.8	.4		
8							.4	.8	.3	
9								.3	.8	.3
10									.3	.8

Note: Due to the strait geometry, the detection capability is also a function of the place where B crosses.

Both A and B would like to use their best strategies. In addition, side A feels that the proportion of boats he is catching now is not sufficient and he would like to catch at least 30% of B's boats. He has two alternatives:

(1) to obtain another detection radar.

(2) to obtain another boat (identical to the present one.)

Since alternative 1, in general, is cheaper, he would like to choose it if possible. Solving the problem (using the P1 formulation) for one boat, we get that the value of the game is .1706. This means that A catches 17.06% of B's boats when he uses the mixed strategy:

$a_1 = .13475$	$a_6 = 0$
$a_2 = .10464$	$a_7 = .17190$
$a_3 = 0$	$a_8 = .08265$
$a_4 = .09289$	$a_9 = .11901$
$a_5 = .12557$	$a_{10} = .16860$

Now if A has another detection unit on board, his new detection capability P^*_{ij} is given by:

(3) $$P^*_{ij} = 1 - (1-P_{ij})(1-P_{ij}) = 2P_{ij} - P^2_{ij}$$

The new detection capability table is shown in Table 2.

TABLE 2

	1	2	3	4	5	6	7	8	9	10
1	.96	.84	.36							
2	.84	.96	.84	.36						
3	.36	.84	.96	.84	.36					
4		.36	.84	.96	.84	.19				
5			.19	.84	.96	.75	.19			
6				.19	.84	.96	.75			
7						.75	.96	.64		
8							.64	.96	.57	
9								.57	.96	.57
10									.57	.96

Solving this game makes the proportion side A catches with two detection units 23.22 per cent. Side A must then modify his strategy to:

$a_1 = .0587$ $\qquad$ $a_6 = .01821$

$a_2 = .20934$ $\qquad$ $a_7 = .22764$

$a_3 = 0$ $\qquad$ $a_8 = 0$

$a_4 = 0$ $\qquad$ $a_9 = .15176$

$a_5 = .18259$ $\qquad$ $a_{10} = .15176$

We see that although the effectiveness of the boat almost doubled, side A's effectiveness increased only by 36%. While the boat is more effective, its detection range did not change. According to the formulation of the problem this effectiveness is not sufficient for A, who would

like to examine the possibility of having two boats in the strait.

Now consider the case where side A has two boats and let $A_{k\ell}$ denote the proportion of time that A assigns boat one to strip k and boat two to strip 1. We have a 100 x 10 matrix if the two boats have different detection capabilities, and a 55 x 10 matrix if they have identical capabilities.

Denote by $P_{(k\ell),j}$ the probability that B is detected when A has boat one at k and boat two at 1 and B is crossing at j. Thus,

$$P_{(k\ell,j} = 1 - (1-P_{kj})(1-P_{\ell j}) \tag{4}$$

Solving this game yields a detection probability of .339 for side A. Therefore, side A can get 33.9 per cent of B's boats if he uses the following mixed strategy.

Assign boat one to strip	boat two to strip	Proportion
1	6	.04388
1	7	.05485
2	7	.11854
2	8	.22673
2	9	.08942
4	9	.03820
5	9	.08235
5	10	.28084
7	10	.06517

Having another identical boat does not double the probability of a catch for side A as expected. The large increase indicates a very poor coverage with a single boat.

D. ZERO SUM GAME WITH SOME KNOWLEDGE OF OPPONENT'S STRATEGY

The nature of a zero sum game is that it has a unique equilibrium. If side A chooses the correct mixed strategy he is guaranteed to get at least the value of the game regardless of what his opponent does. For a zero sum two person game it can be shown that side B cannot get more than minus the value of the game if he plays any strategy when A uses his optimum strategy. Now consider the case where side A has some knowledge of what side B is doing. If side B plays its best strategy, then side A cannot gain by changing his strategy according to the information he has. Side A can gain only if he has information on what side B does and side B does not play according to its best strategy. Obviously, if A has information on time and location of side B shipments, then he intercepts B. On the other hand, he might have some information on probabilities associated with the way B uses some pure strategies. If we have information of the second type we can include this information as constraints in the L.P. formulation [Ref. 4].

1. Example 2

Consider Example 1; and assume that side A has only one ship, but with additional information. He knows that side B is twice as likely to go in strip number 5 than in number 1 or 10. He also knows that less than half the time side B will go through strip number 4, 5, 6, 7. Formulating this as a linear programming problem we have

For A

$\max\ [V' - 1/2\ \omega_3]$

s.t.

$$
(P2)\quad
\begin{array}{lrl}
.8a_1+.6a_2+.2a_3 & -2\omega_1 & \geq V' \\
.6a_1+.8a_2+.6a_3+.2a_4 & & \geq V' \\
.2a_1+.6a_2+.8a_3+.6a_4+.1a_5 & & \geq V' \\
.2a_2+.6a_3+.8a_4+.6a_5+.1a_6+\omega_1 & +\omega_3 & \geq V' \\
.2a_3+.6a_4+.8a_5+.6a_6\quad +\omega_2+\omega_3 & & \geq V' \\
.1a_4+.5a_5+.8a_6+.5a_7 & +\omega_3 & \geq V' \\
.1a_5+.5a_6+.8a_7+.4a_8 & +\omega_3 & \geq V' \\
.4a_7+.8a_8+.3a_9 & & \geq V' \\
.3a_8+.8a_9+.3a_{10} & & \geq V' \\
.3a_9+.8a_{10} & -2\omega_2 & \geq V'
\end{array}
$$

$$\sum_{i=1}^{10} a_i = 1$$

$a_i \geq 0 \quad \omega_3 \geq 0 \quad \omega_1,\ \omega_2$ unrestricted

For B

Min V

s.t.

$$
\begin{array}{ll}
.8b_1+.6b_2+.2b_3 & \le V \\
.6b_1+.8b_2+.6b_3+.2b_4 & \le V \\
.2b_1+.6b_2+8b_3+.6b_4+.2b_5 & \le V \\
.2b_2+.6b_3+.8b_4+.6b_5+.1b_6 & \le V \\
.1b_3+.6b_4+.8b_5+.5b_6+.1b_7 & \le V \\
.1b_4+.6b_5+.8b_6+.5b_7 & \le V \\
.5b_6+.8b_7+.4b_8 & \le V \\
.4b_7+.8b_8+.3b_9 & \le V \\
.3b_8+.8b_9+.3b_{10} & \le V \\
.3b_9+.8b_{10} & \le V \\
-2b_1 \quad +b_5 & = 0 \\
+b_5 - 2b_{10} & = 0 \\
b_4+b_5+b_6+b_7 & \le 1/2
\end{array} \qquad \text{(D2)}
$$

$$\sum_{j=1}^{10} b_j = 1$$

Note that the problems for A and B are dual problems, and the value for both sides must be the same since the game is a zero sum game.

E. LIMITATIONS OF ZERO SUM GAME

The zero sum game formulation to this problem is computationally simple, since equilibrium points exist and the value of the game is unique. There are some limitations, however, that must be considered. By modeling a conflict situation as a zero sum game,one assumes that both opponents see the same payoff function and their preferred alternatives are strictly opposite. In our case of patrol boats and smugglers we assumed some detection function and the payoff was the probability of a patrol boat detecting a smuggler. It is not always true that smugglers know the detection capabilities of the patrol, and in most cases smugglers know very little about this.

The minimax solution of such a game provides an optimal solution against an opponent's optimal strategy, and if the opponent deviates from his "best" strategy the solution does not indicate how to take advantage of his mistake.

The minimax solution was derived assuming complete lack of knowledge about the opponent's strategy in any specific play If this knowledge exists there are better methods which are applicable.

III. DIFFERENT PAYOFF FUNCTIONS

A. DIFFERENT PAYOFF FUNCTIONS

In Chapter 2 we assumed that both side A and side B see the same payoff function which made it a zero sum game. One of the disadvantages of this approach is that side B does not always know the detection capabilities of side A. Thus side B can assume some detection capabilities of side A boats or he may randomize his behavior by having a uniform distribution over all crossing points. If side B chooses to assume some detection capabilities on side A, we can construct a table with entries of the probability that B perceptually will cross successfully in strip j when side A's boat is in strip i. Now each side will observe his probability matrix and try to get the best strategy thinking that his payoff matrix is the true payoff matrix. If we assume that each side will think that the other side knows the "true" payoff matrix each will use a mixed strategy that will guarantee him the value of the game. Therefore, side A will not change his strategy (denote a*). Side B, assuming a payoff matrix P_2, will solve the game and determine a strategy b^1 which is different from b* (strategy of side B when he knows the payoff matrix P_1 of side A). Side A will get a payoff $v^1 = a^{*t} P_1 b^1 \geq a^{*t} P_1 b^*$, where a^t denotes "a transpose."

B. STRATEGY AGAINST OPPONENT WITH RANDOM BEHAVIOR

As stated in section A it is difficult for side B to know the detection capabilities of side A. Without information, side B might assume it is uniform and, therefore, in order to minimize his risk use equal probabilities for each crossing point. Thus, if we have N crossing points, the probability that side B will cross at one of them is 1/N. If side A knows it, he might use this information in order to maximize his expectation. As before we use linear programming to calculate the expected value for side A.

$$\max V' - \frac{1}{N} \sum_{j=1}^{N} \omega_j$$

s.t.

$$\sum_{i=1}^{N} a_1 P_{i_j} + \omega_j - V' \geq 0 \text{ for every } j$$

(P3)

$$\sum_{i=1}^{N} P_i = 1$$

We get this information by recalling that the problem of side A is the dual of that of side B which is

Min V

s.t.

$$\sum_{j=1}^{N} P_{i_j} b_j - V \le 0 \text{ for every } i$$

(D3) $b_j = \frac{1}{N}$

$$\sum_{j=1}^{N} b_j = 1$$

Now we can convert the primal problem to the form which is easily solved by the simplex method by subtracting the first line from the others.

1. <u>Example 3</u>

Consider Example 1, but side A knows that side B intends to use a uniform strategy. The problem then becomes:

$$\text{Max } V - \frac{1}{N} \sum_{j=1}^{N} \omega_j$$

s.t.

$$\begin{array}{ll}
.8a_1+.6a_2+.2a_3 & +\omega_1 \ge V \\
.6a_1+.8a_2+.6a_3+.2a_4 & +\omega_2 \ge V \\
.2a_1+.6a_2+.8a_3+.6a_4+.2a_5 & +\omega_3 \ge V \\
\quad .2a_2+.6a_3+.8a_4+.6a_5+.1a_6 & +\omega_4 \ge V \\
\quad\quad .1a_3+.6a_4+.8a_5+.5a_6+.1a_7 & +\omega_5 \ge V \\
\quad\quad .1a_4+.6a_5+.8a_6+.5a_7 & +\omega_6 \ge V \\
\quad\quad\quad .5a_6+.8a_7+.4a_8 & +\omega_7 \ge V \\
\quad\quad\quad\quad .4a_7+.8a_8+.3a_9 & +\omega_8 \ge V \\
\quad\quad\quad .3a_8+.8a_9+.3a_{10} & +\omega_9 \ge V \\
\quad\quad\quad\quad .3a_9+.8a_{10} & +\omega_{10} \ge V
\end{array}$$

Solving this problem on the IBM 360/67 we get that the payoff is 0.23 which corresponds to side A remaining at strip 3 all the time. The value of the information to side A is .23 - .1706 = .0594, or he increased his efficiency by 35% by having this information about B.

Note

In this case we could solve the problem without using L.P. since we know that side B is equally likely to cross at any point, side A allocates his boat to the strip where he has the largest probability of catching B.

C. CONCLUSION

Whenever side B deviates from his max min strategy, side A is guaranteed to have at least the previous payoff and in most cases he will have more. When side B has wrong ideas about side A detection capabilities, this must reduce his payoff. If side B deviates from his max min strategy (side B uses a uniform strategy) and if side A has this information, then side A can use it to increase his payoff. In general, any information one side has on the other side is very helpful and although we calculate the min max or max min payoff which is the guaranteed payoff, additional information can increase the payoff for side A (or decrease it for him) significantly. Both sides, therefore, can benefit from using information which tells how the opponent deviates from his max min (min max) strategy.

IV. GAME WITH A DEADLINE

We now consider a conflict situation which is limited by time. Such a case arises when smugglers (side B) have perishable items, or intelligence which must be delivered within a certain time period; otherwise they become of no value.

A. GAME WITH A DEADLINE, ONE PATROL BOAT, ONE SMUGGLER BOAT

Denote the period by M (i.e., the smugglers have M nights to cross the strait). Assume that B must cross the strait only once in the period M in order to succeed. Side A (patrol boat) has only one boat and due to limited resources can use it only for k nights, where $k < M$. We assume at least initially that both sides know M and k. Each night both side A and side B make a decision to go to sea or not to go.

Let Γ_k^n denote the value of the game on the n^{th} day before the end of the period. The game matrix is shown in Table 3.

		Side B: go	Side B: no go
Side A	go	v game is over	Γ_{k-1}^{n-1} A has k-1 available days
	no go	-1 game is over	Γ_k^{n-1} A has k available days

TABLE 3

Explanation

(1) If side A and side B both decide to go, side A has some probability of catching side B. This probability can be determined from a zero sum game (see Chapter II) or any other way.

(2) If side B goes but side A does not, B wins and we say the payoff to A is -1.

(3) If side A goes but not side B, then side A loses one available day.

(4) If both do not go, one day is gone from the period and side A has as many available days as before.

For this type of game as shown in Table 3 one can show that the value of the game is given by the following recursive equation [Ref. 10, p. 173]:

$$\Gamma_k^n = \frac{v\ \Gamma_k^{n-1} + \Gamma_{k-1}^{n-1}}{v + \Gamma_{k+1}^{n-1} - \Gamma_{k-1}^{n-1}} \tag{5}$$

with the boundary conditions

$$\Gamma_o^n = -1 \text{ and } \Gamma_n^n = v \text{ , for every } n > 0.$$

Note that for the last period (n=1), the game matrix has the form:

		side B: go	side B: no go
Side A	go	v	1
	no go	-1	1

where $|v| < 1$. By dominance, side B must choose the go

strategy which implies he always chooses to go on the last period if he didn't do it before. Now if side A has n available days he will use them all. Then each day he has a payoff v if side B goes. But since we know that B will go, the value of the game then is v. If side A has more than n available days he can use at most n because he has only one boat.

The solution of the difference equation is given by the following:

<u>Theorem</u>

The solution of the recursive game described above is:

$$\Gamma_k^n = \frac{k(v+1) - n}{n} \tag{6}$$

<u>Proof</u>

From (5) it follows that

$$\Gamma_k^n = \frac{v \cdot \Gamma_k^{n-1} + \Gamma_{k-1}^{n-1}}{v + \Gamma_k^{n-1} - \Gamma_{k-1}^{n-1} + 1} = \frac{\frac{vk(v+1)-(n-1)}{(n-1)} + \frac{(k-1)(v+1)-(n-1)}{n-1}}{1+v+\frac{(k(v+1)-(n-1)}{n-1} - \frac{(k-1)(v+1)-(n-1)}{n-1}}$$

$$= \frac{kv(v+1)-v(n-1)+k(v+1)-(v+1)-(n-1)}{(v+1)(n-1)+(v+1)}$$

$$= \frac{(v+1)[k(v+1)-(n-1)-1]}{(v+1)\ n} = \frac{k(v+1)-n}{n} \qquad \text{q.e.d.}$$

We can, therefore, write the game matrix as:

		Side B	
		go	no go
Side A	go	v	$\frac{(k-1)(v+1)-(n-1)}{n-1}$
	no go	-1	$\frac{k(v+1)-(n-1)}{n-1}$

If side A uses a mixed strategy where x_k^n denotes the probability to go when there are n nights left and side A has k available days remaining, the optimal "go" probability for side A is found by solving:

$$x_k^n v - 1 \cdot (1-x_k^n) = \frac{k(v+1)-n}{n}$$

from which it follows that:

(7) $$x_k^n = \frac{k}{n}$$

Similarly, for side B using mixed strategy with "go" probability y_k^n:

$$y_k^n v + (1-y_k^n)\left(\frac{(k-1)(v+1)-(n-1)}{n-1}\right) = \frac{k(v+1)-n}{n}$$

and therefore:

(8) $$y_k^n = \frac{1}{n}$$

Thus, we conclude that in order to get the value of the game, side A must allocate his available days so the probability that he will go is equal to the ratio of the number of search periods available to A to the total number of remaining periods.

For side B, a uniform probability distribution over the remaining period will give him the value of the game. This probability is independent of the number of available days which side A has.

Knowing the value of the game Γ_k^n, we can calculate the probability that A catches B by

$$P_r\{A \text{ catch } B\} = \frac{\Gamma_k^n + 1}{2}$$

1. Example 4

Consider a game where in any period if side A allocates a boat and side B decides to go, side A gets a payoff of 0.5. There are 10 days left to the end of the period and side A has only six available days.

The strategies for A and B are, to go with probabilities:

$$x_6^{10} = \frac{6}{10} = .6 \text{ and } y_6^{10} = 1/10, \text{ respectively.}$$

The value of the game is

$$\Gamma_5^{10} = \frac{6(1+.5)-10}{10} = -.1.$$

If A and B did not go, we have a new game where

$$v = .5 \quad k = 6 \quad n = 9.$$

Therefore

$$x_6^9 = \frac{6}{9} = .667$$

$$y_6^9 = \frac{1}{9} = .112$$

$$\Gamma_6^9 = \frac{6(1+.5)-9}{10} = 0$$

B. GAME WITH DEADLINE, TWO PATROL BOATS, ONE SMUGGLER BOAT

Consider now the situation where side A has two boats, one capable of doing k_1 and the second k_2 patrol days during the period.

Case 1

Suppose the two boats are identical and side A can allocate his boats according to some optimal plan. If he allocates one boat he gets V_1^* and if he allocates two he gets V_2^*. Side A now has three alternatives and the game is of the form:

	go	no go
two boats	V_2^*	Γ_{n-1}^{k-2}
one boat	V_1^*	Γ_{n-1}^{k-1}
no boat	-1	Γ_{n-1}^{k}

but when side A has only one available day, it becomes:

one boat	V_1^*	-1
no boat	-1	Γ_{n-1}^1

and on the last day of the period:

$$\Gamma_1^k = \begin{cases} V_2^* \text{ if } k \geq 2 \\ V_1^* \text{ if } k = 1 \\ -1 \text{ if } k = 0 \end{cases}$$

Note

(1) The fact that the two boats of side A are identical allows us to lump together the remaining available search period for each boat.

(2) The matrix for Γ_1^k is

	go	no go
two boats	V_2^*	+1
one boat	V_1^*	+1
no boat	-1	+1

Side B always chooses the dominant go strategy.

The solution to this game can be obtained by solving the recursive equation:

(9) $$\Gamma_n^k = f(V_1^* , V_2^*, \Gamma_{n-1}^k, \Gamma_{n-1}^{k-1} , \Gamma_{n-1}^{k-2})$$

where f is the recursive relation, and one must solve a 3 x 2 game. Usually this is done by formulating an LP program, but in this game we find its value graphically by considering the dual problem or:

$$\text{Min } v$$

S.T.

$$V_2^* \, y + \Gamma_{n-1}^{k-2} \, (1-y) \leq v$$

$$V_1^* \, y + \Gamma_{n-1}^{k-1} \, (1-y) \leq v$$

$$-1 \, y + \Gamma_{n-1}^{k} \, (1-y) \leq v$$

where Γ_n^k is the value of the game in period n when side A has a total of k available days.

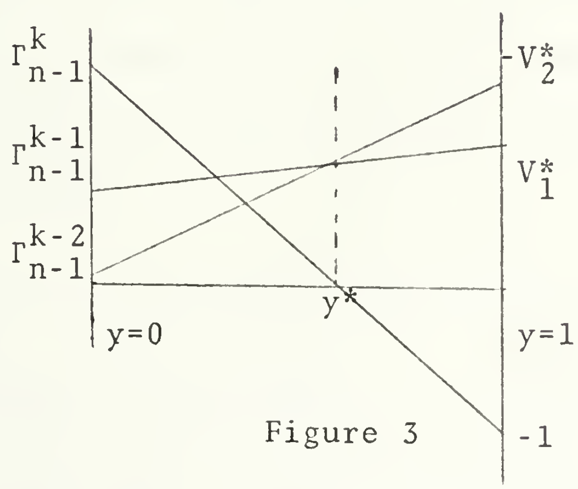


Figure 3

In Figure 3 we see a typical graphic solution. It is always true that $V_2^* > V_1^* > -1$ and

$$\Gamma_{n-1}^{k} \geq \Gamma_{n-1}^{k-1} \geq \Gamma_{n-1}^{k-2}$$

or there could not be dominance.

A solution is the highest point of intersection of two lines or intersection of all three lines or if there are two solutions any convex combination is also a solution.

Although we cannot find an explicit form for Γ_n^k, we can solve the recursive relations for known values of V_1^* and V_2^*.

Case 2

Suppose now that the two boats are not identical. We can define V_1^*, V_2^* and V^{**} as expected payoffs when boat number one, boat number two or both are out according to some optimal allocation.

Let $\Gamma_n^{k_1,k_2}$ be the value of the game played when n periods are left and side A has k_1 available days for boat number one and k_2 for the other boat. This is a 4 x 2 game with matrix:

	go	no go
both boats	V^{**}	$\Gamma_{n-1}^{k_1-1,\ k_2-1}$
boat number one	V_1^*	$\Gamma_{n-1}^{k_1-1,k_2}$
boat number two	V_2^*	$\Gamma_{n-1}^{k_1,k_2-1}$
no boats	-1	$\Gamma_{n-1}^{k_1,k_2}$

and boundary conditions.

$$\Gamma_1^{k_1,k_2} = \begin{cases} V^{**} & \text{if } k_1 \geq 1 \quad k_2 \geq 1 \\ V_1^* & \text{if } k_1 \geq 1 \quad k_2 = 0 \\ V_2^* & \text{if } k_1 = 0 \quad k_2 \geq 1 \\ -1 & \text{if } k_1 = 0 \quad k_2 = 0 \end{cases}$$

$$\Gamma_n^{0,0} = -1$$

where

Γ_n^{0,k_2} and $\Gamma_n^{k_1,0}$ means the corresponding rows must be eliminated from the game matrix.

As stated in Case 1 the value of $\Gamma_n^{k_1,k_2}$ as a function of V^{**}, V_1^*, V_2^* and $\Gamma_{n-1}^{k_1,k_2}$, $\Gamma_{n-1}^{k_1-1,k_2}$ and $\Gamma_{n-1}^{k_1,k_2-1}$ can be found numerically by linear programming or by graphic solution.

Case 3

As a final case we point out the situation where side A has n boats of different types. In principle it is the same problem as Case 2, only the game matrix now is $2^n \times 2$. One has to define all combinations of payoffs and develop a recursive relation by linear programming or graphic methods.

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The purpose of this thesis is to summarize some game theoretical models which can be applied to situations of conflict between two opponents.

Opponent one's objective is to guard a long and narrow strait, with his high speed boats, against the other who tries to cross.

The models considered are zero sum game models, games with different payoff functions and a game with deadline where the

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